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On exterior variational calculus†

R Aldrovandi‡ and R A Kraenkel§

Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405 São Paulo, SP, Brazil

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Abstract. Exterior variational calculus is introduced through examples in field theory. A very simple technique is provided to decide whether or not Lagrangians exist for given sets of field equations and, when they do, to find them. Some applications are made to gauge and chiral fields.

1. Introduction

Exterior differential calculus has been devised as a means to compactify notation, reducing most expressions of tensor analysis to their essentials. In recent years the language of differential forms has become a matter of necessity for theoreticians working in many branches of physics. On the other hand, functional techniques and the closely related variational methods have long been common lore but a systematised exterior functional calculus seems to be missing. Our objective here is to introduce an exterior variational calculus in perfect analogy with differential calculus on finite dimensional manifolds. Previous attempts in this direction (Santilli 1977) have established common points between variational and exterior calculus but some important remaining differences (Anderson and Duchamp 1980) hindered the use of the full power and transparency of the method. The calculus presented below is a complete analogue to exterior differential calculus. Although it may have a sound mathematical basis, it will be introduced here in a purely descriptive way, as a practical device leading quickly to results. In a rather diffuse way, special cases of it have been applied to the study of some specific problems, such as the BRST symmetry (Stora 1984, Zumino *et al* 1984, Faddeev and Shatashvilli 1984) and anomalies (Bonora and Cotta-Ramusino 1983), but its scope is far more general.

This paper is concerned only with 'local' aspects, i.e. properties valid only in some open set in the field functional space. It is our hope, just as happened with differential forms on finite-dimensional manifolds, that functional forms may become of great help in the search for topological functional characteristics. We do not recall here the results of differential calculus. We simply state those which are necessary directly in terms of functional forms, since they are the same. The basic ideas and definitions are presented in § 2. The inverse problem of variational calculus is broached in § 3, where the non-existence of Lagrangians for the Navier-Stokes and the Korteweg-de

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§ Fellow of FAPESP, São Paulo.

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Vries equations is demonstrated in a few lines. The ensuing paragraph shows how to obtain Lagrangians in a systematic way, with examples in non-linear and non-polynomial cases. Applications to gauge theories, including the Wess–Zumino consistency condition and the BRST symmetry, are made in § 5. We finish with a short discussion of some questions involving chiral fields.

2. Preliminary remarks

Let us consider a set of fields $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^N)$ which we suppose to be defined on the spacetime M to fix the ideas, although all that follows is easily adapted to fields defined on other manifolds as some examples will make clear. Each kind of field defines a bundle with M as base space and we shall take local coordinates (x^μ, φ^a) on the bundle. The basic idea is to use the x and the φ as independent coordinates (Anderson and Duchamp 1980). If $\mathscr{L}[\varphi(x)]$ is a Lagrangian density, its total variation under small changes of these coordinates is

$$\delta_{\mathrm{T}}\mathscr{L} = \delta\mathscr{L} + \mathrm{d}\mathscr{L} = (\delta_a\mathscr{L})\delta\varphi^a + (\partial_\mu\mathscr{L})\,\mathrm{d}x^\mu \tag{2.1}$$

where $\delta_a\mathscr{L}$ is a shorthand for $(\delta\mathscr{L}/\delta\varphi^a)$ and $\delta\varphi^a$ is the purely functional variation of φ^a. In the spirit of field theory, $\mathscr{L}$ is supposed to have no explicit dependence on x^μ, so that $\partial_\mu\mathscr{L} = (\delta_a\mathscr{L})\partial_\mu\varphi^a$. Of course, for the part concerned with variations in spacetime (i.e. in the arguments of the fields) we have

$$\mathrm{d}^2\mathscr{L} = \tfrac{1}{2}(\partial_\lambda\partial_\mu\mathscr{L} - \partial_\mu\partial_\lambda\mathscr{L})\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu = 0. \tag{2.2}$$

This is precisely one of the results of exterior calculus which we wish to extend to the φ space. It is natural for the δ operator:

$$\delta^2\mathscr{L} = \tfrac{1}{2}(\delta_a\delta_b\mathscr{L} - \delta_b\delta_a\mathscr{L})\delta\varphi^a \wedge \delta\varphi^b = 0. \tag{2.3}$$

Here $\delta\varphi^a \wedge \delta\varphi^b$ is the antisymmetrisation of the product $\delta\varphi^a\delta\varphi^b$, i.e. just the exterior product of the differentials of the coordinates φ^a and φ^b. In order to enforce the boundary-has-no-boundary property for the total variation, we must impose

$$\delta_{\mathrm{T}}^2 = (\delta + \mathrm{d})^2 = \delta\mathrm{d} + \mathrm{d}\delta = 0 \tag{2.4}$$

but

$$\begin{aligned}\delta_{\mathrm{T}}^2\mathscr{L} &= (\delta\mathrm{d} + \mathrm{d}\delta)\mathscr{L} = \delta\varphi^a \wedge \delta_a(\partial_\mu\mathscr{L}\,\mathrm{d}x^\mu) + \mathrm{d}x^\mu \wedge \partial_\mu(\delta_a\mathscr{L}\delta\varphi^a)\\ &= (\delta_a\partial_\mu\mathscr{L} - \partial_\mu\delta_a\mathscr{L})\delta\varphi^a \wedge \mathrm{d}x^\mu\end{aligned}$$

so that (2.4) requires

$$\delta_a\partial_\mu\mathscr{L} = \partial_\mu\delta_a\mathscr{L}. \tag{2.5}$$

The anticommutation of δ and d implies the commutation of the respective derivatives. This is no novelty, of course, as it happens normally in differential calculus when we separate a manifold into two subspaces. The total variation does not commute with spacetime transformations

$$[\partial_\mu, \delta_{\mathrm{T}}]f = (\partial_\mu\delta x^\lambda)\partial_\lambda f \tag{2.6}$$

but the purely functional variation does. We shall from now on consider only purely functional variations. Furthermore, instead of densities as in the example above, we shall consider only objects integrated on spacetime, such as the action functional

$$S[\varphi] = \int \mathrm{d}^4x\,\mathscr{L}[\varphi(x)]. \tag{2.7}$$

For facility of language, we shall sometimes interchange the terms 'Lagrangian' and 'action'. Differentiating the expression above,

$$\delta S[\varphi] = \int \mathrm{d}^4x\, \delta\mathscr{L}[\varphi(x)] = \int \mathrm{d}^4x \{\delta_a \mathscr{L}[\varphi(x)]\} \delta\varphi^a(x). \tag{2.8}$$

The Fréchet differential of a functional $F[\varphi]$ at a point φ of the functional space and along a direction $\eta(x)$ in that space is defined by

$$F'[\varphi] = \lim_{\varepsilon\to 0} \frac{F[\varphi+\varepsilon\eta]-F[\varphi]}{\varepsilon} = \frac{\mathrm{d}F[\varphi+\pi\eta]}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}. \tag{2.9}$$

$F'[\varphi]$ is a linear operator on $\eta(x)$, and $F'[\eta]=0$ is a linearised version of the equation $F[\varphi]=0$. The integrand (2.8) is the Fréchet derivative of $\mathscr{L}[\varphi]$ along $\eta=\delta\varphi$. The presence of the integration and (2.5) justify the usual procedures of naive variational calculus, such as 'taking variations (in reality, functional derivatives and not differentials) inside the common derivatives' which, allied to an indiscriminate use of integrations by parts (supposing convenient boundary conditions), lends it a great simplicity.

The vanishing of the expression between $\{\}$ in (2.8) gives the field equations, $\delta_a\mathscr{L}[\varphi(x)]=0$. Given a set of field equations $E_a[\varphi(x)]=0$, we shall call its Euler form the expression

$$E[\varphi] = \int \mathrm{d}^4x\, E_a[\varphi(x)]\delta\varphi^a(x). \tag{2.10}$$

The exterior functional (or variational) differential of such an expression will be defined as

$$\begin{aligned}\delta E[\varphi] &= \int \mathrm{d}^4x\, \delta E_a[\varphi(x)] \wedge \delta\varphi^a(x)\\ &= \frac{1}{2}\int \mathrm{d}^4x \{\delta_b E_a[\varphi(x)] - \delta_a E_b[\varphi(x)]\}\delta\varphi^b(x)\wedge\delta\varphi^a(x).\end{aligned} \tag{2.11}$$

The differential of (2.8) is immediately found to be zero, $\delta^2 S[\varphi]=0$. In analogy to the usual 1-forms, 2-forms, etc, of exterior calculus, we shall denote the corresponding functional differentials such as (2.10) and (2.11) as 1-Forms, 2-Forms, etc. A p-Form will be an object like

$$Z[\varphi] = \frac{1}{p!}\int \mathrm{d}^4x\, Z_{a_1a_2\ldots a_p}[\varphi(x)]\delta\varphi^{a_1}(x)\wedge\delta\varphi^{a_2}(x)\wedge\ldots\wedge\delta\varphi^{a_p}(x) \tag{2.12}$$

the exterior product signs indicating a total antisymmetrisation quite analogous to that of differential calculus. A new feature in Forms is that their components in a 'coframe' $\{\delta\varphi^a\}$ as above may be operators, in reality acting on the first $\delta\varphi^a$ at the right. Take, for instance, the Euler Form for a free scalar field

$$E[\varphi] = \int \mathrm{d}^4x[\Box_x + m^2]\varphi_a(x)\delta\varphi^a(x).$$

Its differential will be

$$\delta E[\varphi] = \int \mathrm{d}^4x\{\delta_{ab}[\Box_x+m^2]\}\delta\varphi^a(x)\wedge\delta\varphi^b(x) = 0 \tag{2.13}$$

because the component $\{\delta_{ab}[\Box_x + m^2]\}$ is symmetric. The vanishing of the $\Box$ term may be seen, after integration by parts, as a consequence of $\delta\partial_\mu\varphi^a(x) \wedge \delta\partial^\mu\varphi_a(x) = 0$. The use of operator components provides an automatic extension to the larger space also containing the field derivatives, avoiding the explicit use of jet bundles of rigorous variational calculus (Anderson and Duchamp 1980).

The indices (a_j) in (2.12) are, of course, summed over as they are repeated. To simplify notation, we shall from now on extend this Einstein convention to the spacetime variable x and omit the integration sign, as well as the $(p!)$ factor. Its implicit presence should, however, be kept in mind, as integration by parts will frequently be used. In reality, to make expressions shorter, we shall frequently omit the arguments. Equation (2.10), for example, will be written simply as

$$E = E_a\delta\varphi^a. \tag{2.14}$$

Finally, we shall borrow freely from the language of differential calculus. A Form W satisfying $\delta W = 0$ will be said to be a *closed* Form and a Form W which is a variational differential of another, $W = \delta Z$, will be an *exact* Form.

3. The existence of Lagrangians

The question of the existence of a Lagrangian for a given set of field equations with Euler Form E can then be put in a simple way: is there a 0-Form S as in (2.7) such that $E = \delta S$? Or, when is E locally an exact Form?

Consider expression (2.11) where the $E_a[\varphi(x)]$ are densities just as $\mathscr{L}[\varphi(x)]$ and the differentials appearing are Fréchet differentials

$$\delta E_a[\varphi] = \{\delta_b E_a[\varphi]\}\delta\varphi^b = E'_a[\delta\varphi]. \tag{3.1}$$

The Helmholtz-Vainberg necessary and sufficient condition (Vainberg 1964) for the existence of a local Lagrangian is that, in a ball around φ in the functional space,

$$\varepsilon^a E'_a[\eta] = \eta^a E'_a[\varepsilon] \tag{3.2}$$

for any two increments η, ε. In our notation, with increments ε^a along φ^a and η^b along φ^b, (3.1) tells us that this is equivalent to $\delta_b E_a = \delta_a E_b$, or from (2.11) that

$$\delta E = 0. \tag{3.3}$$

We have here a variational analogue of the Poincaré inverse lemma of differential calculus: for a Form to be locally exact, it is necessary and sufficient that it be closed. In this case, $E_a = \delta_a \mathscr{L}$ for some $\mathscr{L}$. There are, however, equations of physical interest which are not related to an action principle in terms of the fundamental physical fields involved.

Let us look at the notorious case of the Navier-Stokes equation

$$\rho v^j\partial_j v^i + \partial^i p - \mu\partial^j\partial_j v^i = 0 \tag{3.4}$$

which together with

$$\partial_j v^j = 0 \tag{3.5}$$

describes the behaviour of an incompressible fluid of density ρ and coefficient of viscosity μ. The physical fields of interest are the velocity components v^j and the pressure p. The pressure is the obvious candidate for the Lagrange multiplier of the incompressibility condition (3.5), so that we write the Euler Form as

$$E = -(\rho v_j \partial^j v_i + \partial_i p - \mu \partial_j \partial^j v_i)\delta v^i + (\partial_j v^j)\delta p \tag{3.6}$$

with the relative sign conveniently chosen. A direct calculation gives

$$\begin{aligned}\delta E &= -\delta(\rho v_j \partial^j v_i) \wedge \delta v^i - (\partial_i \delta p) \wedge \delta v^i + \mu(\partial_j \partial^j \delta v_i) \wedge \delta v^i + (\partial_j \delta v^j) \wedge \delta p \\ &= -\delta(\rho v_j \partial^j v_i \delta v^i) + \delta p \wedge (\partial_i \delta v^i) - \mu(\partial^j \delta v_i) \wedge (\partial_j \delta v^i) + (\partial_j \delta v^j) \wedge \delta p \\ &= -\delta(\rho v_j \partial^j v_i \delta v^i) \neq 0.\end{aligned} \tag{3.7}$$

The 'offending' non-Lagrangian term can be immediately identified as $\rho v_j \partial^j v_i$. The power of exterior variational calculus is well illustrated in the above few lines, which summarise the large amount of information necessary to arrive at this result (Finlayson 1972). Another example of interest is the Korteweg-de Vries equation, for which the Euler Form is

$$E = (u_t + u u_x + u_{xxx})\delta u \tag{3.8}$$

the indices indicating derivatives with respect to t and x. That no Lagrangian exists may be seen from the simple consideration of the first term in δE, $\delta u_t \wedge \delta u$, which is non-vanishing and cannot be compensated by any other contribution. This example may illustrate an important point: the existence or not of a Lagrangian depends on which field is chosen as the fundamental physical field. Above the chosen field was supposed to be u. In terms of u, no Lagrangian exists. However, a Lagrangian does exist in terms of φ if we put $u = \varphi_x$, as E becomes the closed Form

$$E = (\varphi_{tx} + \varphi_x \varphi_{xx} + \varphi_{xxxx})\delta\varphi. \tag{3.9}$$

When the choice of the fundamental physical field is determined by some other reason, as in quantum field theory, it is of no great help that a Lagrangian may be found by some clever change of variable.

There is an obvious ambiguity in writing the Euler Form for a set of two or more field equations, as multiplying each equation by some factor leads to an equivalent set. Such a freedom may be used to choose an exact Euler Form and to give the Lagrangian a correct sign.

4. Finding Lagrangians

Differential forms have a very convenient local expression which embodies a more complete version of the Poincaré inverse lemma (Warner 1983, Lovelock and Rund 1975, Nash and Sen 1983). We shall state it already adapted to Forms. Let us begin by defining the operation T on the p-Form Z. If

$$Z[\varphi] = Z_{a_1 a_2 \ldots a_p}[\varphi]\delta\varphi^{a_1} \wedge \delta\varphi^{a_2} \wedge \ldots \wedge \delta\varphi^{a_p} \tag{4.1}$$

then TZ is defined as the $(p-1)$-Form given by

$$T[Z] = \sum_{j=1}^{p} (-1)^{j-1} \int_0^1 \mathrm{d}t\, t^{p-1} Z_{a_1 a_2 \ldots a_p}[t\varphi]\varphi^{a_j}\delta\varphi^{a_1} \wedge \ldots \wedge \delta\varphi^{a_j - 1} \wedge \delta\varphi^{a_j + 1} \wedge \ldots \wedge \delta\varphi^{a_p}. \tag{4.2}$$

Under the integration sign, the fields φ^a appearing in the argument of $Z_{a_1 a_2 \ldots a_p}$ are multiplied by the variable t. As t goes from 0 to 1, the field values are continuously deformed from 0 to φ^a. This is a homotopy operation (Nash and Sen 1983) in φ space and T is sometimes called 'homotopy operator'. A more general homotopy $\varphi_t = t\varphi + (1-t)\varphi_0$, with $\varphi_0 \neq 0$ may be used, but without real gain of generality. The important point is that the φ space is supposed to be a star-shaped domain around some 'zero' field (each point may be linked to zero by straight lines). Spaces of this kind are called 'affine' spaces by some authors (Singer 1981). Some important field spaces are not affine. For example, the space of metrics used in general relativity includes no zero; nor does the space of chiral fields with values on a Lie group. For such cases, to be examined later, the use of (4.2) is far from immediate.

The important local expression we have announced states that, for affine functional spaces, Z can always be written as

$$Z = \delta(TZ) + T(\delta Z). \tag{4.3}$$

This result may be obtained from (4.2) by direct verification. A consequence is that a closed Z will be locally exact, $Z = \delta(TZ)$. For a closed Euler Form E, this gives immediately the Lagrangian $\mathscr{L} = TE$, the expression (4.2) reducing to Vainberg's homotopy formula (Atherton and Homsy 1975, Finlayson 1972). Equation (4.3) allows a systematic identification of those pieces of a given E which are Lagrangian derivable and those which are not. This was done directly in (3.7) but (4.3) may be useful in more complicated cases. No term in (3.8) is Lagrangian derivable, since there

$$T\delta E = E \qquad \delta TE = 0. \tag{4.4}$$

If the first term in (3.6) is dropped and the remaining give through (4.2) the Helmholtz-Korteweg Lagrangian

$$\mathscr{L} = p\partial_i v^i - \tfrac{1}{2}\mu(\partial_j v_i)(\partial^j v^i). \tag{4.5}$$

For (3.9), the Lagrangian is immediately obtained

$$\mathscr{L} = \tfrac{1}{2}\varphi\varphi_{tx} + \tfrac{1}{3}\varphi\varphi_x\varphi_{xx} + \tfrac{1}{2}\varphi\varphi_{xxxx}. \tag{4.6}$$

A trivial rule to obtain $\mathscr{L}$ from $E = \delta\mathscr{L} = E_a\delta\varphi^a$ is obtained when E_a is a polynomial in the fields and/or their derivatives: we replace in E $\delta\varphi^a$ by φ^a and divide each monomial of the resulting polynomial by the respective number of fields (and/or their derivatives). A simple example of the use of (4.2) in a non-polynomial theory may be found in Born-Infeld electrodynamics (Born and Infeld 1934). With $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^2 = F_{\mu\nu}F^{\mu\nu}$, its Euler Form is

$$E = \partial^\mu \left(\frac{F_{\mu\nu}}{(1 - F^2/2k)^{1/2}} \right) \delta A^\nu. \tag{4.7}$$

In this case

$$TE = A^\nu \partial^\mu \int_0^1 \frac{\mathrm{d}t\, tF_{\mu\nu}}{(1 - t^2F^2/2k)^{1/2}}$$

gives, after integration and a convenient antisymmetrisation,

$$\mathscr{L} = k[(1 - F^2/2k)^{1/2} - 1]. \tag{4.8}$$

It is sometimes possible, by a clever picking up of terms, to exhibit the Euler Form directly as an exact Form, threby showing the existence and the form of a Lagrangian. Take for Einstein's equations for the pure gravitational field the Euler Form

$$E=(-g)^{1/2}[R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}(R+\Lambda)]\delta g^{\mu\nu} \tag{4.9}$$

for a cosmological constant Λ. We can recognise $\delta(-g)^{1/2}=-\frac{1}{2}(g)^{1/2}g_{\mu\nu}\delta g^{\mu\nu}$ in the second term and separate $\delta R=\delta g^{\mu\nu}R_{\mu\nu}+g_{\mu\nu}\delta R_{\mu\nu}$ to write $E=\delta[(-g)^{1/2}(R+\Lambda)]-(-g)^{1/2}g_{\mu\nu}\delta R_{\mu\nu}$. Of these two terms, the latter is known to be a divergence (Landau and Lifshitz 1975) and the first exhibits the Hilbert-Einstein Lagrangian. The factor $(-g)^{1/2}$ is to be expected if we recall the implicit integration in (4.9). It plays the role of an integrating factor, as E would be neither invariant nor closed in its absence.

5. Gauge fields

The Euler Form for sourceless gauge fields is

$$E=(\partial^\mu F^a{}_{\mu\nu}+f^a{}_{bc}A^{b\mu}F^c{}_{\mu\nu})\delta A_a{}^\nu=(D^\mu F^a{}_{\mu\nu})\delta A_a{}^\nu. \tag{5.1}$$

The coefficient, whose vanishing gives the Yang-Mills equations, is the covariant derivative of the curvature F of the connection A in this same connection. Each component $A^a{}_\mu$ is a variable labelled by the double index (a,μ) and $f^a{}_{bc}$ are the gauge group structure constants. Taking the differential

$$\delta E=(\partial^\mu\delta F^a{}_{\mu\nu}+f^a{}_{bc}A^{b\mu}\delta F^c{}_{\mu\nu}+f^a{}_{bc}\delta A^{b\mu}F^c{}_{\mu\nu})\wedge\delta A_a{}^\nu. \tag{5.2}$$

The last term vanishes if we use the complete antisymmetry (or cyclic symmetry) of $f^a{}_{bc}$: the coefficients become symmetric under the change $(a,\nu)\leftrightarrow(b,\mu)$. Integrating by parts the first term, using again the cyclic symmetry and conveniently antisymmetrising in (μ,ν), we arrive at

$$\delta E=-\tfrac{1}{2}\delta F^a{}_{\mu\nu}\wedge\delta F_a^{\mu\nu}=0. \tag{5.3}$$

The cyclic symmetry used above holds for semisimple groups, for which the Cartan-Killing form is an invariant metric well defined on the group. We have been using such a metric to raise and lower indices since equation (5.1). That no Lagrangian exists in the non-semisimple case has been shown first in the particular case of the Poincaré group (Aldrovandi and Pereira 1986) and then in the general case (Aldrovandi and Pereira 1985) by using the Helmholz-Vainberg theorem. In the semisimple case, (4.2) can be used to obtain

$$\mathscr{L}=\tfrac{1}{2}A_a{}^\nu D^\mu F^a{}_{\mu\nu}=-\tfrac{1}{4}F_a{}^{\mu\nu}F^a{}_{\mu\nu}. \tag{5.4}$$

The space Σ of the A is star-shaped (Singer 1981), so that the integration to get $\mathscr{L}$ is straightforward, $\mathscr{L}$ being valid on the whole Σ as far as no subsidiary condition is imposed. Of course this is not the physical space, which is far more complicated (Wu and Zee 1985). Given the 'large group' Γ, the infinite group formed by all the gauge transformations on spacetime, the physical space is formed by the gauge-inequivalent points of Σ, the quotient space Σ/Γ. Variations on Σ may be locally decomposed into a part 'along' Γ and a part 'orthogonal' to Γ:

$$\delta A_a{}^\nu=\delta^{\parallel}A_a{}^\nu+\delta^{\perp}A_a{}^\nu. \tag{5.5}$$

The part $\delta^{\|}$ parallel to Γ is a gauge transformation. Defined on Σ there are entities which act as representatives of the geometrical entities defined on the gauge group G. Such representatives are, however, dependent on the point in spacetime. The group parameters η^a, in terms of which an element of G is written $g = \exp(\eta^a T_a)$ in some representation generated by (T_a), become fields $\eta^a(x)$. The canonical Maurer-Cartan 1-forms $\omega = g^{-1}\,dg$ on G, written $\omega = \omega^a J_a$ if (J_a) is the basis generating the adjoint representation and satisfying the equations $d\omega = -\omega \wedge \omega$, are likewise represented by fields $\Omega(x) = g^{-1}(x)\delta^{\|}g(x)$, 1-Forms on Γ, whose expression is enough to ensure that Ω satisfies a functional version of the Maurer-Cartan equations,

$$\delta^{\|}\Omega = -\Omega \wedge \Omega \tag{5.6a}$$

or

$$\delta^{\|}\Omega^a = -\tfrac{1}{2} f^a{}_{bc}\Omega^b \wedge \Omega^c. \tag{5.6b}$$

The components Ω^a or the matrix $\Omega_{ij} = \Omega^a (J_a)_{ij} - \Omega^a f^i{}_{aj}$ are alternatively used when more convenient, the same holding for A_μ, $F_{\mu\nu}$, δA_μ, etc.

Gauge subsidiary conditions correspond to 1-Forms along Γ. Take, for example, the one-dimensional Abelian case of electromagnetism, for which the Maxwell Euler Form is

$$E = (\partial^\mu F_{\mu\nu})\delta A^\nu. \tag{5.7}$$

As $\delta A^\nu = \delta^{\|}A^\nu + \delta^{\perp}A^\nu$ and $\delta^{\|}A^\nu = \partial^\mu \delta\eta$ for some parameter field η, an integration by parts shows that the contribution along Γ vanishes. The Lorentz gauge condition is specified by the 1-Form

$$H = \lambda(\partial^\mu A_\mu)\delta\eta = -\lambda A_\mu \partial^\mu \delta\eta = -\lambda A_\mu \delta^{\|} A^\mu. \tag{5.8}$$

The complete Euler Form governing electromagnetism in the Lorentz gauge is consequently

$$E^* = (\partial^\mu F_{\mu\nu})\delta^{\perp}A^\nu - \tfrac{1}{2}\lambda\delta^{\|}(A_\mu A^\mu). \tag{5.9}$$

H is an exact Form only along Γ so that we cannot say that $TH = -\frac{1}{2}\lambda(A_\mu A^\mu)$ is a Lagrangian in the usual sense.

The above considerations can be transposed without much ado to the non-Abelian case. Putting $\delta A_a{}^\nu = D^\nu \delta\eta_a + \delta^{\perp} A_a{}^\nu$ in (5.1), the contribution along the group vanishes again. The Lorentz Form is now

$$\begin{aligned} H = \lambda(\partial^\mu A^a{}_\mu)\delta\eta_a &= -\lambda A^a{}_\mu(D^\mu\delta\eta_a - [A^\mu, \delta\eta]_a) = -\lambda A^a{}_\mu \delta^{\|} A_a{}^\mu \\ &= -\tfrac{1}{2}\lambda\delta^{\|}(A^a{}_\mu A_a{}^\mu). \end{aligned}$$

Supposing a convenient normalisation for the Cartan-Killing metric we have been using implicitly, the total Euler Form may be written

$$E^* = \mathrm{tr}[D^\mu F_{\mu\nu}\delta^{\perp}A^\nu - \tfrac{1}{2}\lambda\delta^{\|}(A_\mu A^\mu)]. \tag{5.10}$$

We have used, for the sector along Γ, the holonomic (or 'coordinate') basis $\{\delta\eta_a\}$, composed of exact Forms. We could likewise have used a non-holonomic basis. With differential forms the choice of basis is in general dictated by the symmetry of the problem. For 1-Forms along Γ, a very convenient basis is formed by the Maurer-Cartan Forms $\{\Omega^a\}$. On the space Σ, such Forms are dual to the 'vector fields'

$$T^a = -D_\mu \frac{\delta}{\delta A_{a\mu}} \tag{5.11}$$

which represent (Faddeev and Shatashvilli 1984) the generators of gauge transformations on Σ. As $\Omega = g^{-1}\delta g = g^{-1}(\delta\eta)g$, a 1-Form $G = G_a\Omega^a$ is related to $K = K_a\delta\eta^a$ simply by $G = g^{-1}Kg$. Now in the basis $\{\Omega^a\}$, $\delta^{\parallel} = \Omega^a T_a$ on 0-Forms, so that, for example,

$$\delta^{\parallel} S = -\Omega_a D_\mu \frac{\delta S}{\delta A_{a\mu}} = -\Omega^a D_\mu E_a{}^\mu = 0 \tag{5.12}$$

with $E_a{}^\mu = D_\nu F_a{}^{\mu\nu}$. This expresses the invariance of S (or of the Lagrangian) under gauge transformations. We can otherwise integrate by parts to obtain

$$0 = \delta^{\parallel} S = (D_\mu \Omega^a) E_a{}^\mu = \delta^{\parallel} A^a{}_\mu E_a{}^\mu \tag{5.13}$$

which again says that the equation is 'orthogonal' to Γ.

The expressions for the gauge anomalies are components of 1-Forms along Γ in the anholonomic basis $\{\Omega^a\}$

$$U = U_a \Omega^a. \tag{5.14}$$

Using (5.6b), we find

$$\delta^{\parallel} U = \tfrac{1}{2}[T_a U_b - T_b U_a - U_c f^c{}_{ab}]\Omega^a \wedge \Omega^b. \tag{5.15}$$

We recognise inside the brackets the expression whose vanishing gives the Wess-Zumino consistency condition, which in this language becomes simply

$$\delta^{\parallel} U = 0. \tag{5.16}$$

Again, U must be locally an exact Form, but only along Γ, so that TU is not a Lagrangian.

Notice that, unlike the case of S in (5.12), the last equation does not express the invariance of U under gauge transformations. Only when acting on 0-Forms does $\delta^{\parallel}$ represent gauge transformations. The situation is again analogous to differential geometry, where transformations are represented by Lie derivatives. Let us consider on Σ objects analogous to the vector fields on manifolds; e.g., entities such as (5.11) or more generally such as $\eta = \eta^a T_a$ or $X = X^a \delta/\delta\eta^a$. Transformations on Forms will be given by the Lie derivatives $L_X = \delta \circ i_X + i_X \circ \delta$, where i_X is the interior product and the symbol $\circ$ stands for composition. For 0-Forms, only the last term remains, but for U the first will also contribute. The invariance of a Form W under a transformation whose generator is represented by a 'Killing field' X will be expressed by $L_X W = 0$. In the case of an Euler Form coming from a Lagrangian, $E = \delta S$, the commutativity between the Lie derivative and the differential operator leads to $L_X E = \delta L_X S$, a well known result: the invariance of $S(L_X S = 0)$ implies the invariance of $E(L_X E = 0)$ but not vice versa. The invariance of E only implies the closedness of $L_X S$ and the equations may have symmetries which are not in the Lagrangian (Okubo 1980).

A final remark concerning gauge fields: we have already used $\delta^{\parallel} A_a{}^\mu = D^\mu \delta\eta_a$. $A_a{}^\mu$ being a 0-Form, this measures to first order its change under a group transformation given by $g(x) = \exp(-\eta(x)) \sim 1 - \delta\eta(x)$. Using $\Omega = g^{-1}(\delta\eta)g$, we can write

$$\delta^{\parallel} A^\mu = D^\mu \Omega. \tag{5.17}$$

A fermionic field will transform according to $\delta^{\parallel}\Psi' = \delta\eta\Psi' = g^{-1}\Omega g \Psi'$, or

$$\delta^{\parallel} \Psi = \Omega\Psi. \tag{5.18}$$

Together with (5.6a), the two last equations express the BRST transformations (Stora 1984, Baulieu 1984), provided the Maurer-Cartan Form Ω is interpreted as the ghost field (Stora 1984, Leinaas and Olaussen 1982) and Slavnov's operator is recognised as $\delta^{\|}$. The well known use of $\delta^{\|}$ to obtain topological results (Mañes *et al* 1985) is a fine illustration of the convenience of variational Forms to treat global properties in functional spaces although it remains, to our knowledge, the only such application so far reported.

6. Chiral fields

We shall finish with a few comments on pure chiral fields, here understood simply as the group-valued fields $g(x)$ encountered above. The functional space reduces to Γ and δ will coincide with the previous $\delta^{\|}$. Neither G nor Γ are star-shaped spaces, so that we must work with tensor fields on the Lie algebra and their functional counterparts. A variation of the Maurer-Cartan form $\omega_\mu = g^{-1}\partial_\mu g$ is the covariant derivative of its corresponding Form:

$$\delta\omega_\mu = -g^{-1}(\delta g)g^{-1}\partial_\mu g + g^{-1}\partial_\mu(gg^{-1}\delta g) = \partial_\mu\Omega + [\omega_\mu, \Omega] = D_\mu\Omega. \qquad (6.1)$$

To obtain the Euler Form corresponding to the two-derivative contribution to the chiral-fields dynamics we start from the usual action

$$S = -\tfrac{1}{2}\operatorname{tr}(\omega_\mu\omega^\mu) \qquad (6.2)$$

from which

$$\begin{aligned} E = \delta S &= -\operatorname{tr}(\omega_\mu\delta\omega^\mu) = -\operatorname{tr}[\omega_\mu(\partial^\mu\Omega + [\omega^\mu, \Omega])] \\ &= -\operatorname{tr}(\omega_\mu\partial^\mu\Omega) = \operatorname{tr}[(\partial^\mu\omega_\mu)\Omega] \qquad (6.3) \\ &= \operatorname{tr}[\partial^\mu(g^{-1}\partial_\mu g)g^{-1}\delta g]. \qquad (6.4) \end{aligned}$$

The existence of a Lagrangian here is a consequence of the functional Maurer-Cartan equation (5.6). In effect,

$$\begin{aligned} \delta E = \delta[(\partial_\mu\omega_a{}^\mu)\Omega^a] &= \delta\omega_a{}^\mu \wedge \partial_\mu\Omega^a + (\partial_\mu\omega_a{}^\mu)\delta\Omega^a \\ &= -(\partial_\mu\Omega^a + f^a{}_{bc}\omega^{b\mu}\Omega^c) \wedge \partial_\mu\Omega_a + \partial_\mu\omega_a{}^\mu)\delta\Omega^a \\ &= (\partial_\mu\omega_a{}^\mu)(\delta\Omega^a + \tfrac{1}{2}f^a{}_{bc}\Omega^b \wedge \Omega^c) = 0. \end{aligned}$$

The presence of Ω in the trace argument in (6.3) would not be evident from the field equation

$$\partial^\mu(g^{-1}\partial_\mu g) = 0. \qquad (6.5)$$

The variation was entirely made in terms of ω_μ and Ω, which belong to star-shaped spaces, and not in terms of $g(x)$. We can consequently follow the inverse way: put (6.4) in the form $E = -\operatorname{tr}(\omega_\mu\delta\omega^\mu)$ and only then use (4.2) to recover (6.2). This is trivial but instructive for the discussion of the five-meson vertex. Let us examine the field equation (Witten 1983):

$$\partial_\mu\omega^\mu + \lambda\varepsilon^{\mu\nu\rho\sigma}\omega_\mu\omega_\nu\omega_\rho\omega_\sigma = 0. \qquad (6.6)$$

The natural extension of (6.3) is the Euler Form

$$E = \operatorname{tr}[(\partial^\mu\omega_\mu)\Omega + \lambda\varepsilon^{\mu\nu\rho\sigma}\omega_\mu\omega_\nu\omega_\rho\omega_\sigma\Omega] \qquad (6.7)$$

which is a Form 'along' Ω. To find the Lagrangian for the second term we have to write it in a holonomic basis while remaining in an affine space. We can then use the group parameter field $\eta(x)$, such that $\omega_\mu = g^{-1}(\partial_\mu \eta)g$ and in terms of which the term is $\lambda\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}(\partial_\mu\eta\partial_\nu\eta\partial_\rho\eta\partial_\sigma\eta\delta\eta)$. The use of (4.2) leads immediately to

$$S = -\tfrac{1}{2}\,\mathrm{tr}[\omega_\mu\omega^\mu - \tfrac{2}{5}\lambda\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\eta\partial_\nu\eta\partial_\rho\eta\partial_\sigma\eta\eta]. \tag{6.8}$$

This has the form of Witten's action. The local considerations above can only give the relative numerical factor between the equation and the Lagrangian.

7. Final comments

We have shown, through many examples, the power of exterior variational calculus in treating some involved aspects of field theories in a very economical way. All cases examined were 'local', i.e. valid in some open set of the field space. Recent years have witnessed an ever growing interest in the global, topological properties of such spaces. Anomalies, BRST symmetry and other peculiarities are now firmly believed to be related to the cohomology of the field functional spaces involved, this belief coming precisely from results obtained through the use of some special variational differential techniques. Many global properties of finite-dimensional manifolds are fairly understood and transparently presented in the language of exterior differential forms. The complete analogy of the infinite-dimensional calculus suggest that, besides being of local interest, it is the natural language in which to examine global properties of field spaces.

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